Unconditional Stability of a Three-Port Network Characterized with S-Parameters

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Abstract —An analytical solution is presented which establishes nine conditions necessary for determining the unconditional stability of a network described with three-port S-parameters. In contrast to the unconditional stability conditions of a two-port network, the unconditional stability conditions of a three-port network are dependent on both the three-port S-parameters and the port terminations. These criteria form the basis for three-port amplifier design, and are used to analyze measured three-port S-parameter data of a silicon BJT at 2.4 GHz.

I. Introduction

THE IDEA OF using three-port S-parameters in the design of high-frequency transistor circuits is not new [1], but only recently have three-port S-parameters been exploited in such designs, particularly in microwave oscillator design [2], [3]. The use of three-port S-parameters in the design of microwave amplifiers has not materialized due to the lack of an analytical criteria for predicting the unconditional stability of such networks.

A typical example of this lack of analytical criteria is found in series-feedback amplifier design. Most series-feedback amplifier designs rely on empirical knowledge for determining the third-port termination which results in the unconditional stability between the input and output ports. Once unconditionally stable, the input and output ports may be simultaneously conjugate matched. One third of the three-port unconditional stability conditions provide this termination information. It is true that analysis employing parameter system conversions can be used in this example, but the use of a three-port S-parameter approach is superior to these methods since parameter conversion can multiply measurement errors up to five times [4]. The significance of measurement error in a design is evident above 12 GHz, where obtaining accurate S-parameters of an active device is difficult.

The use of a three-port unconditional stability criterion is not limited to a feedback-type amplifier, which is actually operated in a two-port sense. It is plausible that truly three-port circuits (circuits which have more than one output port), such as active power-splitters or active circulators [5], can be designed from a three-port approach. Both of these circuits would require a three-port unconditional stability criterion if the port terminations providing the appropriate conjugate match conditions are to be easily determined. Such circuits will be of major importance in silicon and GaAs MMIC's.

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II. The Approach to Unconditional Stability of a Three-Port Network

In the past, two works have dealt with the problem of unconditional stability of a three-port network from a negative-resistance approach. An early work employing three-port Z-parameters [6] established unconditional stability conditions for a two-port network with series feedback. The analysis is based on the envelope of a function [7], which, though mathematically correct, requires a numerical solution. A second work [8] employs three-port S-parameters to establish an unconditional stability criterion. In this work, stability is approached by mapping the S_{11}''-plane and the S_{22}''-plane onto the Γ_3-plane. Unfortunately, two ports must be terminated before any useful information is found, and in doing so, a degree of freedom is lost. In addition, this criterion is based on the two-port unconditional stability "conditions" $|S_{11}|<1$, $|S_{22}|<1$, $K>1$, which Woods [9] has shown to be insufficient.

Unlike these previous works, the present analysis considers stability between two of the ports with a fixed termination on the third port. This approach is advantageous, since it becomes clear that the location of the stability circles, and hence unconditional stability, is dependent on the third-port termination. One also notices that the three-port unconditional stability conditions directly reduce to the form of the two-port unconditional stability conditions when the three-port matrix is degenerated to a two-port matrix.

III. Derivation of the Three-Port Unconditional Stability Conditions

Unconditional stability conditions are found by solving two converse problems. First, the S_{ii}''-plane is mapped onto the Γ_j-plane for a fixed Γ_k. Imposing the requirement that the $|S_{ii}''|\geqslant 1$ region cannot intersect with the $|\Gamma_j|<1$ region leads to the derivation of the stability factors. Second, the Γ_j-plane is mapped onto the S_{ii}''-plane. After imposing the condition that the $|\Gamma_j|<1$ region cannot intersect with the $|S_{ii}''|>1$ region, a second condition is found. For each pair of ports considered for stability, only

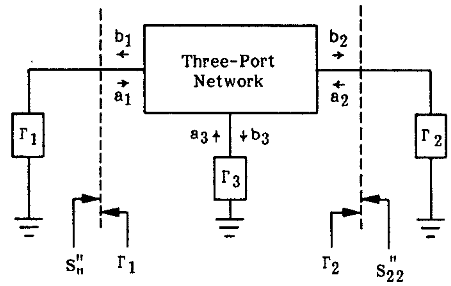

Fig. 1. Configuration of the three-port network characterized with S-parameters.

three conditions (instead of four conditions) arise due to the symmetry of the stability factors.

A. The Stability Factors

In order to find the image of the S_{22}''-plane in the Γ_1-plane, consider the configuration of the three-port network shown in Fig. 1. In a 50-Ω system, it is seen that

$$S_{22}''=\frac{S_{22}-\Delta_{33}\Gamma_1-\Delta_{11}\Gamma_3+\Delta_3\Gamma_1\Gamma_3}{1-S_{11}\Gamma_1-S_{33}\Gamma_3+\Delta_{22}\Gamma_1\Gamma_3} \tag{1}$$

where

$$\Delta_{11}=S_{22}S_{33}-S_{32}S_{23}$$
$$\Delta_{22}=S_{11}S_{33}-S_{31}S_{13}$$
$$\Delta_{33}=S_{11}S_{22}-S_{21}S_{12}$$

and Δ_3 is the determinant of the three-port S-parameter matrix.

Considering Γ_3 fixed, solving for Γ_1, and substituting $S_{22}''=(Z_2-1)\div(Z_2+1)$, it is found that

$$\Gamma_1=\frac{(1-S_{22}-S_{33}\Gamma_3+\Delta_{11}\Gamma_3)Z_2+(\Delta_{11}\Gamma_3-S_{22}+S_{33}\Gamma_3-1)}{(S_{11}-\Delta_{22}\Gamma_3+\Delta_3\Gamma_3-\Delta_{33})Z_2+(\Delta_{22}\Gamma_3-S_{11}+\Delta_3\Gamma-\Delta_{33})} \tag{2}$$

which is a linear fractional transformation between Γ_1 and Z_2. It can be shown that the center and radius of this transformation are given by

$$T=\frac{BC^*+AD^*}{DC^*+(DC^*)^*} \tag{3}$$

$$R=\left|\frac{AD-BC}{DC^*+(DC^*)^*}\right| \tag{4}$$

where

$$A=(1-S_{22}-S_{33}\Gamma_3+\Delta_{11}\Gamma_3) \tag{5}$$
$$B=(\Delta_{11}\Gamma_3-S_{22}+S_{33}\Gamma_3-1) \tag{6}$$
$$C=(S_{11}-\Delta_{22}\Gamma_3+\Delta_3\Gamma_3-\Delta_{33}) \tag{7}$$
$$D=(\Delta_{22}\Gamma_3-S_{11}+\Delta_3\Gamma_3-\Delta_{33}). \tag{8}$$

Substituting (5)–(8) into (3) and (4) leads to

$$T=\frac{C_2^*(\Gamma_3)}{E} \tag{9}$$

$$R=\frac{|J(\Gamma_3)|}{|E|} \tag{10}$$

where

$$C_2^*(\Gamma_3)=\big[(S_{11}^*-\Delta_{33}^*S_{22})+(S_{22}\Delta_3^*-\Delta_{22}^*)\Gamma_3^* +(\Delta_{11}\Delta_{33}^*-S_{11}^*S_{33})\Gamma_3 +|\Gamma_3|^2(S_{33}\Delta_{22}^*-\Delta_{11}\Delta_3^*)\big] \tag{11}$$

$$E=\big[-|\Delta_3|^2|\Gamma_3|^2+|S_{11}|^2+|\Delta_{22}|^2|\Gamma_3|^2-|\Delta_{33}|^2 -2\,\mathrm{Re}[S_{11}\Delta_{22}^*\Gamma_3^*]+2\,\mathrm{Re}[\Delta_3^*\Delta_{33}\Delta_3^*]\big] \tag{12}$$

$$J(\Gamma_3)=(\Delta_{11}\Delta_{22}-S_{33}\Delta_3)\Gamma_3^2 +(\Delta_3+S_{33}\Delta_{33}-S_{11}\Delta_{11}-S_{22}\Delta_{22})\Gamma_3 +(S_{11}S_{22}-\Delta_{33}). \tag{13}$$

Unconditional stability between ports 1 and 2 will be guaranteed when

$$\left|\frac{|J(\Gamma_3)|-|C_2^*(\Gamma_3)|}{E}\right|>1. \tag{14}$$

Rearranging (14) and taking the magnitude squared yields

$$\big||J(\Gamma_3)|-|C_2^*(\Gamma_3)|\big|^2>E^2. \tag{15}$$

Squaring (15) and regrouping the terms, one sees

$$2|J(\Gamma_3)||C_2^*(\Gamma_3)|<|C_2^*(\Gamma_3)|^2+|J(\Gamma_3)|^2-E^2. \tag{16}$$

Squaring (16) yields the expression

$$4|J(\Gamma_3)|^2|C_2^*(\Gamma_3)|^2<\big(|C_2^*(\Gamma_3)|^2+|J(\Gamma_3)|^2\big)^2+E^2 -2E^2\big(|C_2^*(\Gamma_3)|^2+|J(\Gamma_3)|^2\big). \tag{17}$$

Using the identity [10]

$$|C_2^*(\Gamma_3)|^2=|J(\Gamma_3)|^2+EF \tag{18}$$

where

$$F=\big[1-|S_{22}|^2+|S_{33}|^2|\Gamma_3|^2-|\Delta_{11}|^2|\Gamma_3|^2 -2\,\mathrm{Re}[S_{33}\Gamma_3]+2\,\mathrm{Re}[S_{22}^*\Delta_{11}\Gamma_3]\big]$$

and substituting (18) into (17), one sees that

$$4|J(\Gamma_3)|^2\big(|J(\Gamma_3)|^2+EF\big)<\big(2|J(\Gamma_3)|^2+EF\big)^2 +E^4-2E^2\big(2|J(\Gamma_3)|^2+EF\big). \tag{19}$$

After factoring out E^2, (19) reduces to

$$0<\{-4|J(\Gamma_3)|^2+(E^2-2EF+F^2)\}E^2. \tag{20}$$

Further simplification of (20) shows that

$$(F-E)>2|J(\Gamma_3)|. \tag{21}$$

Equation (21) implies that

$$K_3(\Gamma_3)\triangleq\frac{(F-E)}{2|J(\Gamma_3)|}>1 \tag{22}$$

for unconditional stability between ports 1 and 2. A plot of $K_3(\Gamma_3) > 1$, one of the three three-port stability factors, in the Γ_3-plane will show which Γ_3 terminations provide unconditional stability between ports 1 and 2.

Generalizing the port designations, the three-port stability factors are given by

$$K_3(\Gamma_k) = \frac{\left[\left(1-|S_{ii}|^2-|S_{jj}|^2+|\Delta_{kk}|^2\right)+|\Gamma_k|^2\left(|S_{kk}|^2-|\Delta_{ii}|^2-|\Delta_{jj}|^2+|\Delta_3|^2\right)\right.}{2|\left(\Delta_{ii}\Delta_{jj}-S_{kk}\Delta_3\right)\Gamma_k^2+\left(\Delta_3+S_{kk}\Delta_{kk}-S_{ii}\Delta_{ii}-S_{jj}\Delta_{jj}\right)\Gamma_k} \\ \frac{\left.-2\,\mathrm{Re}\left[S_{kk}\Gamma_k\right]+2\,\mathrm{Re}\left[S_{ii}^*\Delta_{jj}\Gamma_k\right]+2\,\mathrm{Re}\left[S_{jj}^*\Delta_{ii}\Gamma_k\right]-2\,\mathrm{Re}\left[\Delta_3^*\Delta_{kk}\Gamma_k^*\right]\right]}{+\left(S_{ii}S_{jj}-\Delta_{kk}\right)|}. \tag{23}$$

Equations (29) and (30) describe the second set of conditions necessary to guarantee unconditional stability of a three-port S-parameter network.

B. The Converse Problem

In order to establish sufficient conditions for stability, the image of the Γ_1-plane must be found in the S_{22}''-plane. In a manner similar to that employed in the derivation of the stability factors, it can be shown that the center and radius of this image are

$$T = \frac{C_1(\Gamma_3)}{F_{21}} \tag{24}$$

$$R = \frac{|J(\Gamma_3)|}{|F_{21}|} \tag{25}$$

where

$$C_1(\Gamma_3) = \left[(S_{22}-S_{11}^*\Delta_{33})+(\Delta_{22}^*\Delta_{33}-S_{22}S_{33}^*)\Gamma_3^* + (S_{11}^*\Delta_3-\Delta_{11})\Gamma_3+|\Gamma_3|^2(\Delta_{11}S_{22}^*-\Delta_3\Delta_{22}^*)\right]$$

$$F_{21} = \left[1-|S_{11}|^2+|S_{33}|^2|\Gamma_3|^2-|\Delta_{22}|^2|\Gamma_3|^2 - 2\,\mathrm{Re}[S_{33}\Gamma_3]+2\,\mathrm{Re}[S_{11}^*\Delta_{22}\Gamma_3]\right].$$

In order to guarantee stability in this view,

$$|T|+|R| \leqslant 1$$

or, equivalently,

$$|T| \leqslant 1-|R|. \tag{26}$$

Since $|T|$ could equal zero, it must be guaranteed that $|R| \leqslant 1$. Consequently, $|S_{22}''| < 1$ for all $|\Gamma_1| < 1$ when

$$F_{21} > |J(\Gamma_3)|. \tag{27}$$

Similarly solving for where the Γ_2-plane maps into the S_{11}''-plane, one sees that $|S_{11}''| < 1$ for all $|\Gamma_2| < 1$ when

$$F_{12} > |J(\Gamma_3)|. \tag{28}$$

Generalizing (27) and (28), it is found that

$$F_{ij} > |J(\Gamma_k)| \tag{29}$$

$$F_{ji} > |J(\Gamma_k)| \tag{30}$$

where

$$F_{ij} = 1-|S_{jj}|^2+|S_{kk}|^2|\Gamma_k|^2-|\Delta_{ii}|^2|\Gamma_k|^2 - 2\,\mathrm{Re}[S_{kk}\Gamma_k]+2\,\mathrm{Re}[S_{jj}^*\Delta_{ii}\Gamma_k]$$

$$F_{ji} = 1-|S_{ii}|^2+|S_{kk}|^2|\Gamma_k|^2-|\Delta_{jj}|^2|\Gamma_k|^2 - 2\,\mathrm{Re}[S_{kk}\Gamma_k]+2\,\mathrm{Re}[S_{ii}^*\Delta_{jj}\Gamma_k]$$

$$J(\Gamma_k) = (\Delta_{ii}\Delta_{jj}-S_{kk}\Delta_3)\Gamma_k^2 + (\Delta_3+S_{kk}\Delta_{kk}-S_{ii}\Delta_{ii}-S_{jj}\Delta_{jj})\Gamma_k + (S_{ii}S_{jj}-\Delta_{kk}).$$

IV. Summary of the Conditions

Stability between two of the ports has been investigated. However, all three driving-point S-parameters must be considered for three-port stability. It has been shown that for Γ_3 values which result in $K_3(\Gamma_3) > 1$, all passive loads on port 2 will result in stability at port 1. It is not necessarily true that all Γ_2 terminations result in stability between ports 3 and 1. In order to find the subset of Γ_2 terminations which result in stability between ports 3 and 1, $K_3(\Gamma_2)$ is found and plotted as a function of fixed Γ_2. If Γ_2 and Γ_3 are chosen from the respective set where $K_3(\Gamma_3) > 1$ and $K_3(\Gamma_2) > 1$, then for those terminations $|S_{11}''| < 1$. Similar reasoning will show that to guarantee $|S_{22}''| < 1$, it is required that both $K_3(\Gamma_1) > 1$ and $K_3(\Gamma_3) > 1$ for some pair of Γ_1 and Γ_2 terminations. Consequently, to guarantee that $|S_{ii}''| < 1$ for $i = 1, 2, 3$, simultaneously, Γ_1, Γ_2, and Γ_3 must belong to the subset of terminations where $K_3(\Gamma_1) > 1$, $K_3(\Gamma_2) > 1$, and $K_3(\Gamma_3) > 1$, respectively.

Table I summarizes the possible combinations of i, j, and k for (23), (29), and (30). The nine equations represented in Table I establish the conditions necessary for three-port unconditional stability. These equations state which passive terminations result in passive driving-point S-parameters for all ports. Explicitly, the requirements for three-port unconditional stability are

$$\text{i)}\ K_3(\Gamma_1) > 1 \quad F_{23} > |J(\Gamma_1)| \quad F_{32} > |J(\Gamma_1)| \tag{31a}$$

$$\text{ii)}\ K_3(\Gamma_2) > 1 \quad F_{13} > |J(\Gamma_2)| \quad F_{31} > |J(\Gamma_2)| \tag{31b}$$

$$\text{iii)}\ K_3(\Gamma_3) > 1 \quad F_{12} > |J(\Gamma_3)| \quad F_{21} > |J(\Gamma_3)| \tag{31c}$$

for some set of terminations $\Gamma_1, \Gamma_2, \Gamma_3$. If any one of these equations is violated, then the network is conditionally stable in the three-port sense for this set of terminations.

V. Analysis of Measured Three-Port S-Parameters

The three-port S-parameters of an NE46734 transistor biased at $V_{CE} = 10$ V and $I_C = 50$ mA were measured with an automatic network analyzer at 2.4 GHz. An error-correcting scheme [10] using the indefinite property for this three-terminal device (with base port 1, emitter port 2, and collector port 3) was used to obtain the following S-

TABLE I
COMBINATIONS OF i, j, AND k FOR EQUATIONS (23), (29), AND (30)

i	j	k
2	3	1
3	2	1
1	3	2
3	1	2
1	2	3
2	1	3

TABLE II
PREDICTED COMMON-COLLECTOR S-PARAMETERS FROM THE THREE-PORT MATRIX AND MEASURED COMMON-COLLECTOR S-PARAMETERS

QUANTITY	MEASURED VALUES	CALCULATED VALUE*	ABSOLUTE ERROR MAGNITUDE	PHASE
S'_{11}	0.728 $\angle -164.69°$	0.688 $\angle -172.61°$	0.040	7.92°
S'_{12}	0.773 $\angle -36.76°$	0.790 $\angle -40.27°$	0.017	3.51°
S'_{21}	1.081 $\angle -105.95°$	1.037 $\angle -109.92°$	0.044	3.97°
S'_{22}	0.420 $\angle 63.12°$	0.468 $\angle 55.25°$	0.048	7.87°

*Γ_3 = 0.998 $\angle 176.94°$

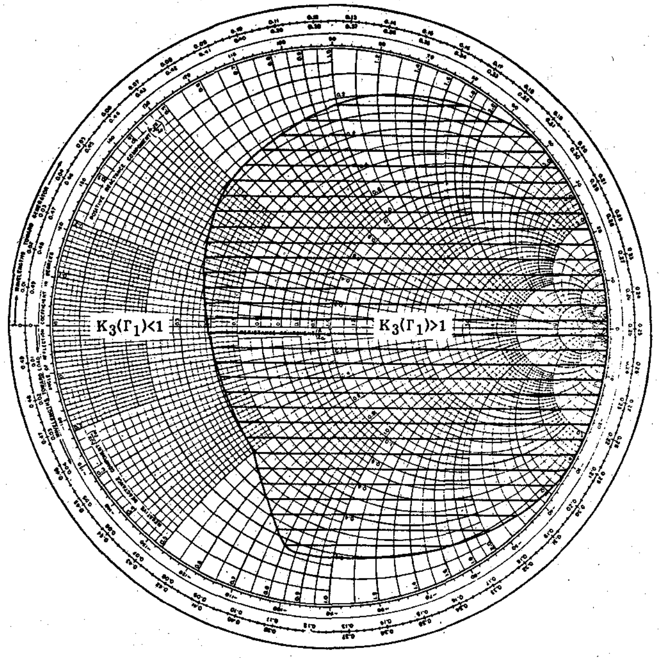

Fig. 2. Stability between ports 2 and 3 viewed as a function of Γ_1. (Crosshatched area represents values of Γ_1 where $K_3(\Gamma_1) > 1$, $F_{23} > |J(\Gamma_1)|$, $F_{32} > |J(\Gamma_1)|$.)

parameters:

$$\begin{bmatrix} 0.306\angle 135.09° & 0.824\angle -35.53° & 0.526\angle 10.56° \\ 0.659\angle -74.01° & 0.466\angle 55.53° & 0.526\angle 10.90° \\ 1.052\angle 14.53° & 0.061\angle -60.24° & 0.346\angle -102.28° \end{bmatrix}.$$

In order to check the validity of these error-corrected S-parameters, the common-collector two-port S-parameters were measured and compared with the corresponding two-port S-parameters found from terminating the collector port with a short circuit ($\Gamma_3 = 0.998\angle 176.94°$). The corresponding data of Table II show that good agreement exists between the measured and calculated two-port S-parameters.

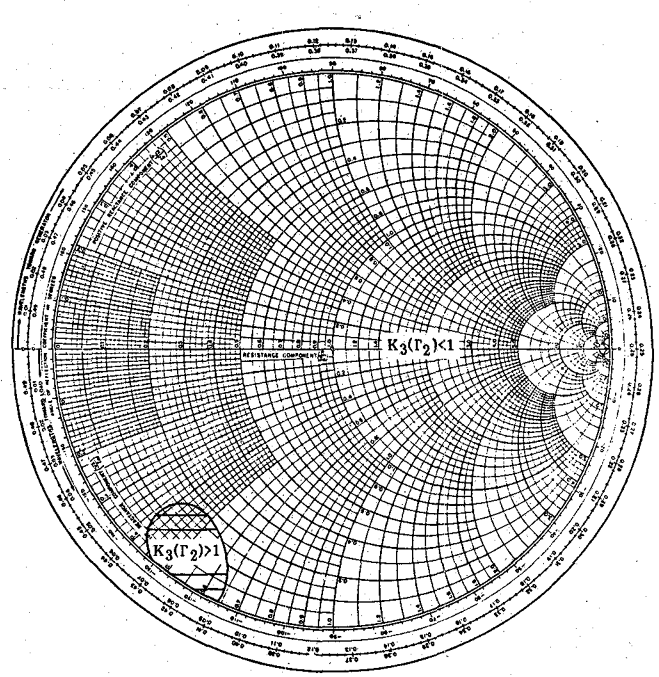

Fig. 3. Stability between ports 3 and 1 viewed as a function of Γ_2. (Crosshatched area represents values of Γ_2 where $K_3(\Gamma_2) > 1$, $F_{31} > |J(\Gamma_2)|$, $F_{13} > |J(\Gamma_2)|$.)

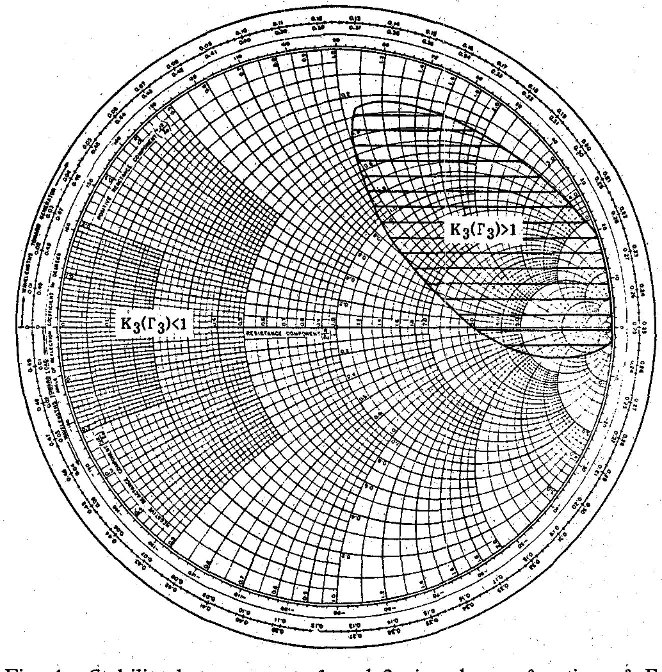

Fig. 4. Stability between ports 1 and 2 viewed as a function of Γ_3. (Crosshatched area represents values of Γ_3 where $K_3(\Gamma_3) > 1$, $F_{12} > |J(\Gamma_3)|$, $F_{21} > |J(\Gamma_3)|$.)

Γ-plane plots of Γ_1, Γ_2, and Γ_3 are shown in Figs. 2, 3, and 4, respectively. These plots show which terminations satisfy conditions i), ii), and iii) respectively, in Section IV.

Any such set of $\Gamma_1, \Gamma_2, \Gamma_3$ which satisfies i), ii), and iii) will guarantee that the three-port is unconditionally stable.

In addition, if one wishes to determine two-port series-feedback configurations, Figs. 2–4 give vital termination information. For example, a capacitive termination in the emitter (port 2) results in unconditional stability between ports 3 and 1 (see Fig. 3). In Fig. 2, one sees that feedback over a large area in the base lead results in unconditional stability between ports 2 and 3. The designer may choose a particular termination in these areas after considering gain or noise requirements.

VI. Conclusions

An analytical solution has been presented which establishes conditions for the unconditional stability of a network described with three-port S-parameters. It has been shown that nine conditions, dependent on the terminations, must be satisfied to guarantee that $|S_{ii}''| < 1$ for $i = 1, 2, 3$. Measured data have been analyzed and plotted, thus providing the designer with a graphical representation of the type of termination (capacitive, inductive, etc.) necessary to provide three-port unconditional stability. These plots also provide stability information for all possible two-port configurations.

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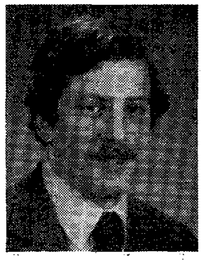

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